Homework 6

Joe Puccio

October 4, 2012

Exercise 3.2.2

(a) The limit points of B are -1 and 1.

(b) No, B is not a closed set because it does not contain -1 and 1 which are limit points.

(c) No, B is not an open set because $\forall \epsilon > 0\ \exists x \mid V_\epsilon(x) \cap B \nsubseteq B$. In fact, in this instance $\forall \epsilon > 0\ \nexists x \mid V_\epsilon(x) \cap B \subseteq B$ which is equivalent to $\forall \epsilon > 0\ \forall x \mid V_\epsilon(x) \cap B \nsubseteq B$.

(d) As seen from the statement above, all points are isolated points. So, yes.

(e) $\overline{B} = B \cup \{\text{-}1, 1\}$.

Exercise 3.2.3

(a) Neither. All points in $\mathbb{Q}$ are isolated because $\mathbb{R}$ is dense in $\mathbb{Q}$. That is, there does not exist an ϵ neighborhood for any point in $\mathbb{Q}$ such that every point in this neighborhood is in $\mathbb{Q}$. This is essentially because between any two rational numbers, an irrational can be found. This shows that $\mathbb{Q}$ cannot be open. $\mathbb{Q}$ cannot be closed because irrational numbers are limit points of $\mathbb{Q}$ and they are not in $\mathbb{Q}$.

(b) $\mathbb{N}$ is closed because it has no limit points which is due to the fact that every element in $\mathbb{N}$ is isolated. It is not open also because every element in it is isolated.

(c) This set is not closed because 0 is a limit point of the set and the set does not contain 0. This set is open.

(d) Neither. 0 is a limit point that is not in the set which means that $(0, 1]$ is not closed. And the element 1 in the set does not have any ϵ neighborhoods such that the neighborhood is entirely in the set because the right hand side of the ϵ neighborhood is always going to be outside of the set.

(e) Neither. This set is not closed because we see that 2 is a limit point and 2 is not in the set (it's the limit of the sequence). It is not open because each point is isolated. In fact, we can explicitly define its isolation by saying that $\forall n$, choosing an ϵ neighborhood such that $\epsilon \leq \frac{1}{(n+1)^2}$ then this neighborhood will not contain any points of the set and therefore the term n will be an isolated point.

Excercise 3.2.4

Proof. We are given that $(a_n) \to x$, which is equivalent to saying that $\forall \epsilon > 0 \; \exists N \in \mathbb{N} \mid \forall n \geq N \; |a_n - x| < \epsilon$ which means that we can say that the sequence $a_n \in V_\epsilon(x)$ and because $a_n \subseteq A$ by our own assumptions, we can make the statement that

$$\forall V_\epsilon(x) \;\; a_n \in A \cap V_\epsilon(x) \equiv \text{x is a limit point.}$$

$\square$

Excercise 3.2.9

(a) By using the definition of a limit point,

$$\forall \epsilon > 0 \; V_\epsilon(y) \cap (A \cup B) = (V_\epsilon(y) \cap A) \cup (V_\epsilon(y) \cap B)$$

Looking at the right side of the equation, we can quite easily see that y must be the limit point of either A or B or both.

(b) Let the set of the limit points of a set Y be denoted L_Y. Then we have:

$$\begin{aligned} \overline{A \cup B} &= A \cup B \cup L_A \cup L_B \\ &= A \cup L_A \cup B \cup L_B \\ &= \overline{A} \cup \overline{B} \end{aligned}$$

(c) Yes.